Quant - Modelling Joint Distributions

Introduction

Industrial enterprises need to import raw materials and export produced goods from time to time. As an easy example of this you can consider an airline service provider 'AeroTours' from the country 'Wonderland' where due to unavailability of natural resources, aviation fuel cannot be produced. The operating costs of such a company are heavily dependent on the price of oil in 'Petroland', where oil is produced and also the currency exchange rate that applies between Wonderland and Petroland, among other factors. Therefore, to be able to manage the operating costs or to boost profitability, AeroTours may choose to enter into financial derivatives contracts. Using such contracts they may try to mitigate their probable losses or even make speculations about asset prices, exchange rates to seek profit.

In this challenge you would explore and price some financial contracts, commonly referred to as 'derivatives' which enterprises like AeroTours can utilize for their risk management problems.

'Derivatives'[1] are contracts which derive their value from the prices of some other assets (e.g, stock, oil, etc). These assets are sometimes also referred to as 'underliers'.

Simple Financial Derivatives

One of the simplest examples of derivatives is 'Options'[2]. An option is a contract which offers the buyer the rights, but not an obligation, to buy or sell an asset (e.g, stock, oil) at a previously agreed upon price (called 'Strike') on a predertermined date (called 'Expiration').

An option where such rights to buy (sell) the underlying asset are offered is called as a 'call' ('put') option. To give an example, if the price of the underlying asset were X_T on the expiration date T then, a call option with Strike K will allow the buyer to buy the asset at the Strike price K. Thus, if $X_T \geq K$ then the buyer will exercise the call option and buy one unit of the asset at price K instead of X_T thereby making profit of $X_T - K$. However, if $X_T < K$ the option is not exercised and expires worthless. Similar reasoning holds for the Put options with the inequalities and the signs reversed. The price of entering a call option contract can then be represented mathematically as,

$$C(X, K, T) = \mathbb{E}[max(X_T - K, 0)]$$

$$\implies C(X, K, T) = \int_0^\infty max(X_T - K, 0) f_X(X_T) dX_T$$

where,

1. $C(X, K, T)$ - price of call option on asset X with Strike as K and expiration at T.
2. X_T - price of the asset X as of time T
3. $f_X(X_T)$ - probability density function of time T asset price X_T.

This implies that the buyer of the call option profits if the price of the asset at time T is above the strike price K but his/her losses are limited since he/she would not exercise the option in the case where $X_T < K$.

Options contracts are one of the simplest ways in which one can speculate on the values of asset prices for profit-seeking or risk management purposes. However, market participants (enterprises like Aerotours in our case) often at times find it more suitable to enter into contracts with more complex characteristics owing to reasons like affordability of entering such specific contracts. In the remainder of this challenge you would explore some of these 'more complex' contracts also commonly known as 'exotics'.

Questions

To price the the payoffs in the questions 1 and 2 below you need to use the set of option prices given in the input data set. The file 'Input Data' contains the strikes and the corresponding call option prices for different underliers, the tab 'Oil Call Option Prices' contains the data for the underlier 'Oil' and the tab 'FX Call Option Prices' contains the data for the underlyer 'Exchange Rate'. Throughout this problem, it is assumed that the cost of borrowing money from today to time T is zero. The prices provided are significant upto 6 digits and the results provided should also be significant upto 6 digits.

Pricing complex derivatives

1. Sometimes it is desired by market participants to have payoffs which pay one unit of a certain currency if the asset level $X_T \geq K$. Such options are commonly called as 'Digital Options'. The payoff for these contracts is mathematically represented as,

$$D(K, T; X) = \mathbb{E}[\mathbb{I}(X_T > K)]$$

where, $\mathbb{I}(X_T > K)$ is 1 only if $X_T > K$ and 0 otherwise. $D(K, T; X)$ denotes the price of the contract defined on asset X with strike K and expiration at time T. Evaluate the price of this contract for both the assets i.e. Oil and Exchange Rate for the strikes given in the file 'Output' and enter them correspondingly in the column 'Price'. For underlier Oil the strikes are provided in the tab 'Oil Digital' and for the underlier 'Exchange Rate' strikes are provided in the tab 'FX Digital'. The set of call options prices are provided in the file 'Input Data'.

2. In this case the payoff function is mathematically represented as,

$$P(K, T; X) = \mathbb{E}[(\text{Max}(X_T - K, 0))^2]$$

$P(K, T; X)$ denotes the price of the contract on asset X with strike K and expiration at time T. Evaluate the price of this contract for both the assets i.e. Oil and Exchange Rate for the strikes given in the file 'Output'. For underlier Oil the strikes are provided in the tab 'Oil Exotic' and for the underlier 'Exchange Rate' strikes are provided in the tab 'FX Exotic'. The set of call options prices are provided in the file 'Input Data'.

3. So far, the payoffs you priced involved only one asset price. In this problem, you will price payoffs which depend on the time T asset prices of both the assets X_1, X_2[Asset $X_1(T)$ is Oil and asset $X_2(T)$ is Exchange Rate.]. These payoffs are represented mathematically as follows,

$$Q_1(X_1(T), X_2(T); B_2) = \mathbb{E}[X_1(T) \times \text{Max}(B_2 - X_2(T), 0)]$$

$$Q_2(X_1(T), X_2(T); B_1) = \mathbb{E}[X_2(T) \times \text{Max}(X_1(T) - B_1, 0)]$$

Since these payoffs depend on the prices of both X_1, X_2 using the provided prices for these in the input data sets you can design a method to model the joint distribution for the asset prices $(X_1(T), X_2(T))$. [The prices of contracts $Q_1(X_1(T), X_2(T); B_2)$ and the corresponding strikes (B_i) are given in the file 'Input Data' in the tab 'Joint_FX_Put' and the prices of contracts $Q_2(X_1(T), X_2(T); B_1)$ are given in the tab 'Joint_Oil_Call'of the file 'Input Data'.] Such kinds of contracts allow investors to get exposure to a certain asset conditional upon the event that the price for a different asset is above/below a specified level. In the case of the airliner in our problem, this kind of situation may arise if Aerotours wants to take a view on the prices of oil conditioned upon the exchange rate being above/below a certain strike level (which is captured by the parameters B_is in above payoffs). Using your model of the joint distribution built from the above set of prices, you are requested to compute the prices for the payoff represented mathematically as follows,

$$F(X_1(T), X_2(T); B_1, B_2) = \mathbb{E}[\text{Max}(X_1(T) - B_1, 0) \times \text{Max}(B_2 - X_2(T), 0)]$$

Evaluate this payoff for the set of (B_1, B_2) pairs provided in the tab 'OilCall_FXPut' in the file 'Output' and enter them correspondingly in the column 'Price'. Again, to provide some insight, such contracts are entered by investors like Aerotours when they want to limit their risk exposure to both future oil prices as well as future currency exchange rates.

*Please note that the joint distribution obtained in question 3 should also recover the input data used to solve questions 1 and 2 i.e. 'Oil Call Option Prices' and 'FX Call Option Prices'.

Input Data

http://cdn.hackerrank.com/contests/gsquantify2016/Input_Data.xlsx

Output

http://cdn.hackerrank.com/contests/gsquantify2016/Output.xlsx

Model Doc & Source Code Submission

Please upload the following in a single zip file:

1. The file 'Output' with answers, significant upto 6 digits, populated as explained above.(Other than entering the prices in the column 'Price' of each tab, please do not make any other changes to the file.)
2. All source code files zipped in a folder with all relevant files (Quote any references you may have used. Code with well written comments is desired. Choice of programming language is not a constraint.)
3. A documentation of your solution. This can be in PDF, PPT, PPTX, DOC or DOCX formats. Do include the following aspects in your model doc: a) Assumptions (if any); b) Any mathematical simplifications/approximations; c) Modelling choices and comments on appropriateness of answers.

Note:

1. The model document should be electronically legible. Scanned copies of handwritten documents will not be evaluated.
2. Ensure consistency between the output generated by the code and the results presented in the output file.
3. The methodology implemented in the code and the methodology explained in the model document should be consistent.
4. If there are any additional ideas, modelling approaches that you wish to include in the documentation which could not be implemented, please clearly mark them so.

Evaluation Criteria:

Your submission will be evaluated based on the numerical accuracy of the solutions as well as the approach chosen to solve the problems.

**The decision of the moderators is final and binding on any disputes during the competition as well as in the evaluation phase.

References

1. https://en.wikipedia.org/wiki/Derivative_(finance)
2. https://en.wikipedia.org/wiki/Option_(finance)